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BAND ASYMPTOTICS ON LINE BUNDLES OVER S^2

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1. Introduction

Let $E \to M$ be a Hermitian line bundle over a compact Riemannian manifold M. The choice of a connection, **a**, on E which is compatible with the Hermitian structure determines a Bochner-Laplace operator, $\Delta_{\mathbf{a}}$, acting on the sections of E. If $q \in C^\infty(M)$, we can form the Schrödinger operator $\Delta_{\mathbf{a}} + q$ and consider its selfadjoint extension to $L^2(E)$. The objective of this paper is to study some spectral properties of this operator in the special case when M is the standard 2-sphere. This problem has been studied recently by Ruishi Kuwabara, [3], who showed that if the curvature of the connection is an odd 2-form the spectrum of the Schrödinger operator forms "bands" of fixed width about the eigenvalues of the Laplacian associated to the SO(3)-invariant connection. In this paper we sharpen Kuwabara's results and describe the asymptotic distribution of eigenvalues in the bands, thus generalizing a theorem of Weinstein's [5], in the flat case.

Added in proof. The referee has alerted us to another paper by Kuwabara which has appeared recently (Math. Z. **187** (1984) 481–490). In this paper asymptotic distributions of eigenvalues are obtained for connections in which the curvature is *not* odd (in which case there is no clustering). The result described in §4 can be regarded as a second order refinement of this result in the same sense that [1] is a second order refinement of [5]

2. Preliminaries

The set of isomorphism classes of line bundles over S^2 is indexed by the integers, the indexing map being the first Chern class followed by the canonical isomorphic $H^2(S^2, \mathbf{Z}) \cong \mathbf{Z}$. More precisely, for every $m \in \mathbf{Z}$ there is an essentially unique Hermitian line bundle with Chern number m, $E_m \to S^2$, and

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such that the natural action of SO(3) on S^2 lifts to an action on E_m which is unitary on the fibers. In fact, under the usual isomorphism $S^2 \cong \mathbf{CP}^1$, E_1 is the hyperplane bundle, and the various E_m are tensor powers of E_1 and its dual.

For every m, the set X of connections on E_m which are compatible with the Hermitian structure is naturally an affine space over the vector space $\mathfrak{A}^1(S^2)$ of real-valued 1-forms on S^2. Furthermore, there is a distinguished connection on E_m, which we will denote by $\mathbf{a}_0$. This is the unique SO(3)-invariant connection, with curvature $\Omega_m = \mathrm{i}m\Theta/2$, Θ being the natural volume form of S^2. $\mathbf{a}_0$ is also the unique connection with a harmonic curvature form and, from the complex point of view, the $(1,0)$ connection of E_m. It is natural to take $\mathbf{a}_0$ as the origin in X and thus establish a bijection $X \cong \mathfrak{A}^1(S^2)$.

Let Δ_m denote the Bochner-Laplace operator associated with the harmonic connection $\mathbf{a}_0$. Our first task is to describe the Laplacian of an arbitrary connection as a perturbation of Δ_m.

2.1. Lemma. *Let $E \to M$ be a Hermitian line bundle over the Riemannian manifold M, and let $\mathbf{a}_0$, $\mathbf{a}_1$ be any two connections on E compatible with the Hermitian structure. Let $\beta = \mathbf{a}_1 - \mathbf{a}_0$, $\beta \in \mathfrak{A}^1(M)$, be the 1-form carrying $\mathbf{a}_0$ to $\mathbf{a}_1$, and let Δ_j be the Bochner-Laplace operator corresponding to $\mathbf{a}_j, j = 1, 2$. Then*

$$\Delta_1 = \Delta_0 + \frac{2}{i}\nabla^0_{\hat\beta} + \frac{1}{i}\operatorname{div}\hat\beta + |\beta|^2. \tag{2.1}$$

Here $\hat\beta$ denotes the vector field on M metric-dual to β, and ∇^0 denotes covariant differentiation with respect to $\mathbf{a}_0$.

Proof. Let $U \subset M$ be an open set over which there is a section e of E which has constant unit length. Let α be the real 1-form describing covariant differentiation ∇^0 in the frame e, that is,

$$\nabla^0(fe) = (df + i\alpha f)e$$

for all $f \in C^\infty(U)$. It is not hard to see that then

$$\Delta_0(fe) = \left[\Delta(f) + (2/i)\langle df, \alpha\rangle + \left(i\delta(\alpha) + |\alpha|^2\right)f\right]e, \tag{2.2}$$

where Δ is the Laplace-Beltrami operator on M, δ: $\mathfrak{A}^1(M,\mathbf{C}) \to \mathfrak{A}^0(M,\mathbf{C})$ is the adjoint of exterior differentiation, and $\langle\ ,\ \rangle$ denotes the pairing given by the Riemannian metric. Replacing α by $\alpha + \beta$ in (2.2) and using the fact that $\delta(\beta) = -\operatorname{div}\hat\beta$, we obtain (2.1). q.e.d.

The previous lemma shows how the Laplacian of an arbitrary connection on E_m is a first order perturbation of the Laplacian Δ_m associated with the harmonic connection. It can be shown (see [2]) that the spectrum of Δ_m consists of the eigenvalues

$$\lambda_k = \left(k + \frac{|m|+1}{2}\right)^2 - \frac{m^2+1}{4}, \qquad k = 0, 1, 2, \cdots,$$

with multiplicity $2k + 1 + |m|$. As Kuwabara shows in [3], for connections with an odd curvature form the above multiple eigenvalues split into bands of bounded width. We will now describe the asymptotic distribution of eigenvalues in these bands.

3. The averaging method

Let $\beta \in \mathfrak{A}^1(S^2)$, $q \in C^\infty(S^2)$, and consider the operator

$$Q = (2/i)\nabla_{\hat{\beta}}^0 + (1/i)\operatorname{div}\hat{\beta} + |\beta|^2 + q.$$

By Lemma 2.1, $\Delta_{\mathbf{a}} + q = \Delta_m + Q$, where $\mathbf{a}$ is the translate of the harmonic connection by β. Let

$$A_m = \left(\Delta_m + \frac{m^2+1}{4}\right)^{1/2};$$

this is a first order, elliptic, selfadjoint pseudodifferentiation operator (ΨDO) with principal symbol $H_0(X, \zeta) = |\zeta|$, the Riemannian norm. Instead of working with $\Delta_{\mathbf{a}} + q$, we will consider $(A_m^2 + Q)^{1/2}$.

3.1. Lemma. *Let $R = \frac{1}{4}(A_m^{-1}Q + QA_m^{-1})$. Then the ΨDO, S, defined by*

$$\left(A_m^2 + Q\right)^{1/2} = A_m + R + S$$

is of order (-1) and its principal symbol is $-\frac{1}{2}H_0^{-3}(\hat{\beta})^2$.

Here by $\hat{\beta}$ we mean the function on T^*S^2 given by contraction with the vector field $\hat{\beta}$. The proof of Lemma 3.1 is practically identical to the proof of Lemma 4.5 in [4], and will be omitted.

We now apply the averaging method to the operator $(A_m^2 + Q)^{1/2}$; see [4, §6]. Let

$$A^0 = A_m - \frac{|m| + 1}{2}$$

and $U(T) = \exp it A^0$. This is a 1-parameter group of Fourier integral operators which is 2π-periodic since the spectrum of A^0 consists of the nonnegative integers. For every ΨDO, P, on E_m define

$$P^{av} = \frac{1}{2\pi}\int_0^{2\pi} U(t)PU(-t)\,dt,$$

$$\tau(P) = \frac{1}{2\pi i}\int_0^{2\pi} dt \int_0^t U(s)PU(-s)\,ds.$$

By Egorov's theorem P^{av} and $\tau(P)$ are ΨDO of the same order as P. By Lemma 6.3 of [4], we have

3.2. Lemma. *With the notation of Lemma 3.1, $(A_m^2 + Q)^{1/2}$ is unitarily equivalent to*

$$A_m + R^{av} + \tfrac{1}{2}[\tau(R), R]^{av} + S^{av},$$

modulo operators of order (-2).

Squaring, we obtain

3.3. Corollary. *$\Delta_{\mathbf{a}} + q$ is unitarily equivalent to*

$$\Delta_m + Q^{av} + \tfrac{1}{4}A_m^{-2}(Q^{av})^2 + A_m[\tau(R), R]^{av} + 2A_m S^{av},$$

modulo operators of order (-1).

Q^{av} is generally of order one, so the "band" phenomenon does not occur. However, in certain cases Q^{av} is of order zero. Let ϕ_t denote the Hamiltonian flow in $T^*S^2-\{0\}$ associated with the Hamiltonian H_0. Given any $f \in C^\infty(T^*S^2-\{0\})$, let

$$f^{av} = \frac{1}{2\pi}\int_0^{2\pi} \phi_t^* f\, dt.$$

We will denote the Poisson bracket of functions on T^*S^2 by $\{\ ,\ \}$.

3.4. Proposition. *Q^{av} is of order zero if and only if the curvature, Ω, of the connection* $\mathbf{a}$ *is odd, i.e., $\iota^*\Omega = -\Omega$, where ι is the antipodal map. In that case, $\Delta_{\mathbf{a}} + q$ is unitarily equivalent to an operator of the form $\Delta_m + B$, where B is a zeroth order operator with principal symbol*

$$(3.1)\quad \left(|\beta|^2\right)^{av} - H_0^{-2}\left(\hat{\beta}^2\right)^{av} - \frac{H_0}{2\pi}\int_0^{2\pi} dt \int_0^t \left\{\phi_t^*\hat{\beta}H_0^{-1}, \phi_s^*\hat{\beta}H_0^{-1}\right\} ds + q^{av},$$

which Poisson commutes with H_0.

Proof. Q^{av} is of order zero if and only if $(\beta)^{av} = 0$, that is, iff $\int_\gamma \beta = 0$ for all geodesics $\gamma \subset S^2$. By Proposition 3.3 in [3], this is equivalent to the existence of $f \in C^\infty(S^2)$, $\beta' \in \mathfrak{A}^1$ such that $\beta = \beta' + df$ and $\iota^*\beta' = \beta'$. This implies that $\iota^*\Omega = -\Omega$. Conversely, if Ω is odd, an easy application of Stokes' theorem shows that $\int_\gamma \beta = 0$ for all oriented geodesics γ, and so Q^{av} is of order zero.

If Ω is odd, the principal symbol of Q^{av} is simply $(|\beta|^2 + q)^{av}$, because the subprincipal symbol of $(2/i)\nabla_\beta^0 + (1/i)\mathrm{div}\,\hat{\beta}$ vanishes (apply equation A4 of [1]). By the Corollary 3.3 and Lemma 3.1, the symbol of B is

$$\left(|\beta|^2 + q\right)^{av} - H_0^{-2}\left(\hat{\beta}^2\right)^{av} + H_0\sigma,$$

where σ is the symbol of $[\tau(R), R]^{av}$. It is not hard to show that

$$\sigma = \frac{-1}{2\pi}\int_0^{2\pi'} dt \int_0^t \left\{\phi_t^* r, \phi_s^* r\right\} ds,$$

where r is the principal symbol of R, i.e., $H_0^{-1}\hat{\beta}$.

4. Band asymptotics

We keep the notation of the previous section and assume that the curvature Ω of the connection $\mathbf{a}$ is odd. As we mentioned earlier, the spectrum of Δ_m consists of the eigenvalues

$$\lambda_k = \left(k + \frac{|m|+1}{2}\right)^2 - \frac{m^2+1}{4}, \qquad k = 0,1,2,\cdots,$$

with multiplicity $d_k = 2k+1+|m|$ (see [2]). By the second part of Proposition 3.4 and standard max-min arguments, there is a constant C such that the spectrum of $S = \Delta_{\mathbf{a}} + q$ is contained in $\bigcup_{k=0}^{\infty}(\lambda_k - C, \lambda_k + C)$. In fact, it is not hard to see that for large k there are precisely d_k eigenvalues of S in $(\lambda_k - C,\ \lambda_k + C)$, counting multiplicities. Denote such eigenvalues by $\mu_1^{(k)},\cdots,\mu_{d_k}^{(k)}$ and let

$$\lambda_j^{(k)} = \lambda_k - \mu_j^{(k)}, \qquad j = 1,\cdots,d_k.$$

Let Z be the unit cosphere bundle of S^2 and let dv denote the SO(3)-invariant measure on Z with total mass equal to one. We can now state our main result.

4.1. Theorem. *The sequence of measures* $\{\nu_k\}$, *given by*

$$d_k\nu_k(\lambda) = \sum_{j=1}^{d_k} \delta\left(\lambda - \lambda_j^{(k)}\right),$$

converges weakly to the measure $F(\beta, q)_* \, dv$, *where* $F(\beta, q)$ *is the restriction of the function* (3.1) *to* Z.

Wih Proposition 3.4 at hand, the proof is identical to that of Weinstein, [5], in the scalar case.

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